of suggestions and additions.

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THE USE OF THE METHOD OF AVERAGING TO STUDY NON-LINEAR OSCILLATIONS OF THE CELTIC STONE*

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The approximate solution of the equations of the perturbed motion of a celtic stone near its position of equilibrium, obtained in /1/ by retaining in these equations terms of the second order with respect to the perturbations and averaging** (**In connection with a footnote in /3/ which appeared later than /1/, we note that the solution obtained in /3/ is identical with that appearing in /1/) is used for a qualitative and quantitative explanation of the following effect established by numerical integration of the complete equations /2/. If the celtic stone is rotated about the vertical axis in a specified direction, then after a fairly short time it ceases to rotate, begins to oscillate about the horizontal axis, and then resumes its rotation in the opposite direction. For some of the models of the celtic stone the change in the direction of rotation may occur more than once.

Let m be the mass of the body, $Gx_1x_2x_3$ the coordinate system attached to the body whose axes are directed along the principal central axes of inertia of the body, A, B, C are the corresponding moments of inertia, x_0, y_0 are the horizontal coordinates of the centre of mass in the fixed coordinate system $O_0x_0y_0z_0$ (the $O_0x_0y_0$ plane is the same as the reference plane), φ, ψ, θ are the Euler angles determining the orientation of the system $Gx_1x_2x_3$ relative to $O_0x_0y_0z_0$, ξ, η, ζ are the coordinates of the point I of contact of the body with the reference plane in the system $Gx_1x_2x_3$, ρ_1, ρ_2 are the radii of curvature of the body at the point J with coordinates $(0, -a, 0)$ in the system $Gx_1x_2x_3$, α is the angle defining the position of the principal axes of curvature at the point J relative to the axes Gx_1, Gx_2. We study the same model of the celtic stone as that in /2/. In particular, the following inequalities hold for this model:

$$\rho_2 > \rho_1 > a, \quad 0 < \alpha < \pi/2, \quad ma\rho_1 < \bar{A} + \bar{C} - B < ma\rho_2$$
$$m\rho_1\rho_2 > B > \bar{A} > \bar{C} \quad (\bar{A} = A + ma^2, \ \bar{C} = C + ma^2)$$

The system under consideration represents a non-holonomic Chaplygin system. Its equations of motion are independent of the angle ψ and admit of the family of solutions

$$\theta = \pi/2, \ \varphi = 0, \ \psi^{\cdot} = \omega \tag{1}$$

where ω is an arbitrary constant. The solutions correspond to uniform rotation of the body about the vertical axis Gx_2, and to the equilibrium state of the body when $\omega = 0$. The point of contact I of the body with the plane coincides with the point J of the body.

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The stability of this motion was studied in detail in /4/. Here we shall consider the case of instability, when $\omega < \omega_0$

$$\omega_0^2 = \frac{mg(\rho_1 - a)(\rho_2 - a)}{(\bar{A} + \bar{C} - B)(\rho_1 + \rho_2 - 2a) + ma(a\rho_1 + a\rho_2 - 2\rho_1\rho_2)}$$

The approximate solution of the equations of motion has the form /1/

$$\theta - \frac{\pi}{2} = \frac{1}{\sqrt{\bar{A}}}\left[B_1 \exp\left(\frac{N_1}{2}\psi\right)\cos bt \cos\gamma - B_2 \exp\left(\frac{N_4}{2}\psi\right)\cos dt \sin\gamma\right] \tag{2}$$

$$\varphi = \frac{1}{\sqrt{\bar{C}}}\left[B_1 \exp\left(\frac{N_1}{2}\psi\right)\cos bt \sin\gamma + B_2 \exp\left(\frac{N_4}{2}\psi\right)\cos dt \cos\gamma\right]$$

Here $b, d\ (b > d > 0)$ and B_1, B_2 denote the frequencies and initial amplitudes of the normal oscillations of the body about the horizontal axes, and γ is the angle defining the direction of these axes. The values of the parameters b, d, γ were given in /1/.

Assuming that at the initial instant $\psi = 0$, the precession of the body is given by the equation

$$B\psi^{\cdot 2} = H(\psi)$$

$$H(\psi) = B\omega^2 + b^2 B_1^2 (1 - \exp(N_1\psi)) + d^2 B_2^2 (1 - \exp(N_4\psi))$$

$$N_1 = -Nb^2, \quad N_4 = Nd^2, \quad N = \frac{a}{g}\frac{\cos\gamma\sin\gamma}{\sqrt{\bar{A}\bar{C}}}(\bar{A} - \bar{C}) > 0$$

The coefficients N_1, N_4 are functions of $\gamma, b^2, d^2\ (N_1 < 0, N_4 > 0)$ /1/. Eqs.(2) determine the normal oscillations of the body, and these oscillation do not decay. When $\psi > 0$, oscillations of frequency d predominate, while when $\psi < 0$, we have the frequency b.

Inspecting the function $H(\psi)$ we find that it vanishes at two values, ψ_1 and ψ_2, of opposite signs; let us specify that $\psi_1 > 0$ and $\psi_2 < 0$. Then the precessional motion will represent a periodic variation in the angle ψ between the values ψ_1 and ψ_2 at which the rotation changes its direction. Thus when the initial angular velocity ω is small, the direction of rotation of the celtic stone will change irrespective of the direction of the initial rotation.

When the initial angular velocity is positive(negative), the instant t_1 (instant t_2) of the first change of direction is given by the formula

$$t_1 = \int_0^{\psi_1} \frac{d\psi\sqrt{B}}{\sqrt{H(\psi)}} \qquad \left(t_2 = \int_{\psi_2}^{0} \frac{d\psi\sqrt{B}}{\sqrt{H(\psi)}}\right)$$

The initial amplitudes of the normal oscillations of the body are determined by the shift of the point of contact I from the point J at the initial instant. Let the point I deviate at the initial instant by ξ_0 in the direction of the Gx_1 axis, and by ζ_0 in the direction of the Gx_3 axis. The quantities ξ_0 and ζ_0 are assumed to be small compared with the size of the body. Using the above displacements, we can find the initial perturbations u_0, v_0 of the variables θ and φ, and the initial amplitudes of the normal oscillations

$$B_1 = -\xi_0\Delta\sin\gamma\sqrt{\bar{C}}\left(a + \frac{\bar{A}}{mg}d^2\right) + \zeta_0\Delta\cos\gamma\sqrt{\bar{A}}\left(a + \frac{\bar{C}}{mg}d^2\right)$$

$$B_2 = -\xi_0\Delta\cos\gamma\sqrt{\bar{C}}\left(a + \frac{\bar{A}}{mg}b^2\right) - \zeta_0\Delta\sin\gamma\sqrt{\bar{A}}\left(a + \frac{\bar{C}}{mg}b^2\right);\ \Delta = \frac{1}{\rho_1\rho_2}$$

Let us apply the results obtained to the model of the celtic stone studied in /2/. The parameters of the body are as follows: $m = 150$ g, $A = 4.5 \times 10^5$ g$\cdot$mm^2, $B = 6\cdot 10^5$ g$\cdot$mm^2 $C = 2\cdot 10^5$ g$\cdot$mm^2 $a = 10$ mm, $\rho_1 = 25$ mm, $\rho_2 = 500$ mm and $|\omega| \leqslant 10$ sec^{-1}. The critical angular velocity is $\omega_0 \approx 32$ sec^{-1}.

We shall consider the neighbourhood of the position of equilibrium. Let the initial perturbations u_0, v_0, ω be of the order of the small parameter ε. The order of magnitude of the initial amplitude of oscillations of the body is given by

$$\eta = \max\{|B_1|/\sqrt{\bar{C}}, |B_2|/\sqrt{\bar{C}}\}$$

The quantity η is a dimensionless parameter depending on α. The deviation of the point I from J is chosen to be the same as in /2/, i.e. $\xi_0 = 0$, $\zeta_0 = 2$ mm.

The initial velocity ω must be small compared with the characteristic velocity such as e.g. the critical velocity ω_0. We shall estimate the order of smallness of this initial velocity in terms of the dimensionless parameter $\varepsilon = \omega/\omega_0$. Henceforth, we shall assume that $|\omega| = 1$ sec^{-1}, so that $\varepsilon = 3.2\cdot 10^{-2}$.

The figure shows the dependence of the parameter η on α. We see that when $10° \leqslant \alpha \leqslant 30°$ the numerical value of η is of the same order as ε.

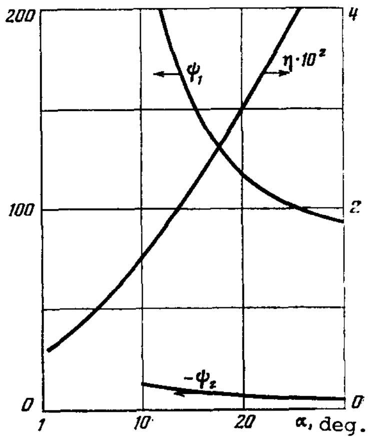
When $10° \leqslant \alpha \leqslant 30°$ and $|\omega| = 1$ sec^{-1}, the approximate values of the roots ψ_1 and ψ_2 of the equation $H(\psi) = 0$ can be found from the formulas

$$\psi_1 \approx \frac{1}{N_4} \ln \frac{1+q+r}{r}, \quad \psi_2 \approx \frac{1}{N_1} \ln \frac{1+q+r}{q}, \quad q = \frac{b^2 B_1^2}{B}, \ r = \frac{d^2 B_2^2}{B}$$

and we obtain the following estimates for t_1 and t_2:

$$\psi_1 (1+q+r)^{-1/2} < t_1 < 2\psi_1, \quad -\psi_2 < t_2 < -2\psi_2 \tag{3}$$

The figure shows the dependence of the roots ψ_1 and ψ_2 on α. For a given value of α the value of ψ_1 is greater than that of $|\psi_2|$ in all cases. The estimates (3) show that for a given value of α t_1 is greater than t_2.



Thus we can conclude that the direction of rotation changes much more rapidly if the body rotates in the positive direction, and this agrees with the results of numerical experiment /2/ where $\alpha = 30°$, $|\omega| = 1$ and the interval of integration was 4 min. The first experiment, where $\alpha = 5°$, $|\omega| = 5$ and the interval of integration was 30 sec., confirms this. The authors discovered the change in the direction of rotation only in the case when the initial angular velocity was positive, because the interval of integration shorter than t_1. Had they used a longer interval of integration, e.g. 2 min, they would have noticed that the change in direction occurs for either direction of the initial angular velocity.

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THE SUFFICIENT CONDITIONS FOR THE EXISTENCE OF ASYMPTOTICALLY PENDULUM-LIKE MOTIONS OF A HEAVY RIGID BODY WITH A FIXED POINT*

A.Z. BRYUM and G.V. GORR

The present paper continues the study of the asymptotically pendulum-like motions (APM) of a heavy rigid body begun in /1/, where a specific mass distribution is not assumed here a priori. The first Lyapunov method /2/ is used to obtain new sufficient conditions for the existence of APM, which cannot be described by the well-known particular solutions of the Euler-Poisson equations.

1. The equations of the first approximation. We shall attach to the body a special coordinate system /3/ and assume that the centre of mass lies in the principal plane of the ellipsoid of inertia constructed for the fixed point. Then the equations of motion will have the form /3/

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